Sustainable Monetary Policy and Inflation Expectations

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Abstract

I show that the short-term nominal interest rate can anchor private-sector expectations into low inflation—more precisely, into the best equilibrium reputation can sustain. I introduce nominal asset markets in an infinite horizon version of the Barro-Gordon model. I then analyze the subset of sustainable policies compatible with any given asset price system at date $t = 0$. While there are usually many sustainable inflation paths associated with a given set of asset prices, the best sustainable inflation path is implemented if and only if the short-term nominal bond is priced at a certain discount rate. My results suggest that policy frameworks must also be evaluated on their ability to coordinate expectations.

1 Introduction

Inflation in advanced countries has been low and stable for the last twenty years. Remarkably, central banks have been able to bring inflation under control while preserving their discretion at policymaking. Independence, transparency, and an emphasis on price stability are usually credited for the improved inflation record. It is important that we have a comprehensive understanding of how the current framework enables successful monetary policy, especially if additional institutional changes are being considered.

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Most theoretical analysis of the monetary policy framework builds upon the literature on commitment and discretion pioneered by Kydland and Prescott (1977) and Barro and Gordon (1983b). The initial premise is that there is a short-run benefit from unexpected inflation. If monetary policy is discretionary, the optimal policy is not compatible with rational expectations and the result instead is an inflation bias. The institutional design is then modeled as modifying the policy objectives, either directly or through an inflation contract, in a way that reins in the ex-post incentives to inflate by the monetary authority.[1]

An old observation is that reputation can substitute for commitment: agents may forgo short-term benefits in order to secure cooperation over the long term. Reputation arises naturally in infinitely repeated games.[2] Barro and Gordon (1983a) show that low inflation can be sustained with trigger strategies in an infinitely repeated game between the monetary authority and the private sector. Formally, an inflation path is said to be sustainable (or enforceable) if it belongs to a sub-game perfect equilibrium. Much of the subsequent work has focused on the best sustainable policy.[3] Unfortunately infinitely repeated games have multiple equilibria, and some may be worse than the no-reputation equilibrium. Rogoff (1987) writes, "*Repeated game models replace a cooperation problem with a coordination problem.*" Torsten Persson and Guido Tabellini (1994) emphasize the "*problematic*" aspect of a "*serious multiple-equilibrium problem;*" while Thomas Sargent (1999) writes that "*the multitude of outcomes mutes the model empirically and undermines the intention of early researchers to use the rational expectations hypothesis to eliminate parameters describing expectations.*"[4]

This paper puts forward the hypothesis that the monetary authority can coordinate private-sector expectations into low inflation—more precisely, into the best equilibrium reputation can sustain. I show that the short-term nominal interest, determined at spot markets at the initial date, can anchor private-sector expectations to the lowest sustainable inflation at all dates and contingencies. Formally, I start with an infinitely repeated version of the

[1] This approach started with the celebrated "conservative" central banker proposed by Rogoff (1985). Walsh (1995) first formalized inflation contracts. The subsequent literature is very large. Other strands of the literature emphasize learning or compare the performance of different Taylor rules. See Sargent (1999) and Clarida, Gali and Gertler (2000), respectively.

[2] This is not the only way to think about reputation, though. Barro (1986) and Backus and Driffill (1985), among others, show how uncertainty about the policymaker "type" allows for a different notion of reputation: the "good" policymaker strives to keep inflation low to distinguish himself from the "bad" policymaker. See Rogoff (1987) for a survey.

[3] An incomplete list is Ireland (1997), Stokey (2002) and Atkeson and Kehoe (2005).

[4] The quotations are from Persson and Tabellini, eds (1994), p.9., and Sargent (1999), p. 38. The equilibrium multiplicity is sometimes regarded as a strength of the theory, as in Chari, Christiano and Eichenbaum (1998).

economy in Barro and Gordon (1983a) and characterize all the sustainable equilibria.[5] Then I introduce a date $t = 0$ spot market for nominal and real assets. By design, no action in the asset markets changes the set of sustainable equilibria—in other words, asset markets leave the cooperation problem intact.[6] I show that the mapping between asset prices and sustainable policies is usually a correspondence, i.e., for a given asset price there are usually either many or no sustainable inflation paths compatible with an equilibrium. The exception, though, is of the upmost interest: there is a one-to-one mapping between the best sustainable policy and the highest discount price for the short-term nominal bond at date $t = 0$.

This result is made possible by three key properties of the best sustainable equilibrium. First, the inflation rate is a function of the exogenous state of the economy along the equilibrium path. Second, at all dates the inflation rate is the lowest of all sustainable inflation rates. Third, the only sustainable inflation path with the lowest inflation rate at date $t = 1$ is the best sustainable equilibrium. These properties are closely related to the bang-bang structure of the best sustainable equilibrium, documented in Abreu et al. (1990).

Thus the coordination problem has moved from the very large space of possible inflation paths to a single spot market that central banks are very familiar with. While a formal treatment is beyond the scope of this paper, coordination on the short-term nominal bond market seems possible. Central banks have a good hold of the short-end of the nominal yield curve, through their role in primary markets, and perhaps enhanced by transparency. Crucially for the theory, coordination should only be possible when agents' beliefs are heterogeneous—otherwise the central bank could build up reputation at will, rendering it useless in the first place.

I briefly sketch how coordination on the short-term nominal interest rate may work. I introduce a stage with multiple rounds of "fictitious play" before date $t = 0$ markets open.[7] In the fictitious play state the monetary authority auctions the short-term nominal bond, loosely resembling the primary market for U.S. Treasury bonds. Each round, agents learn about other agents' beliefs. If agents' initial beliefs are sufficiently heterogeneous, private-sector expectations converge to the highest discount price for the short-term nominal interest rate as the number of rounds of fictitious play increases.[8] Interestingly, the coordination does not hinge on the monetary authority's foresight of the very same theory presented here.

[5]For this I rely heavily on the work by Abreu, Pearce and Stacchetti (1990).

[6]Thus I purposely avoid the mechanism described in Calvo (1978), for example.

[7]Fictitious play assumes that agents do not try to influence the future play of their opponents—see Fudenberg and Kreps (1993) and Fudenberg and Takahashi (2008) and references therein.

[8]In a celebrated experiment, Huyck, Battalio and Beil (1993) found that auctioning the right to play led to coordination on the payoff-dominant equilibrium. See Crawford and Boseta (1998) for a discussion of the theoretical underpinnings.

Equally important, the central bank cannot coordinate on the best sustainable policy if private-sector beliefs are already coordinated on some other equilibrium. This is crucial for the theory: reputation is only as valuable as it is costly to lose it. If the monetary authority could coordinate expectations into the best sustainable policy at any time, the resulting game is renegotiation proof and the only resulting equilibrium would be the one-period Nash equilibrium.

Finally, I also show that, in order to uniquely pin down the best sustainable inflation path with a credible announcement, the monetary authority would need to commit to a state-contingent plan for inflation in the first period. Such commitment is actually sufficient to implement the optimal monetary policy.

There are two important limitations to my analysis. First, I focus on how reputation can *lower* the inflation rate. The one-to-one mapping with the short-term nominal interest rate fails, for example, if the best sustainable inflation rate is *higher* than in the Nash equilibrium. Similarly, I have to assume that the zero nominal interest rate bound is not binding. Second, learning dynamics are decoupled from the model, so it is not immediately clear how to evaluate past or future institutional changes. It also remains daunting to evaluate the theory on the basis of the observed inflation path.

Section 2 presents an overview of the main result in the paper. Section 3 lays out the concept of sustainable monetary policy and derives the properties of the best sustainable policy. Asset markets are then introduced in Section 4 and the mapping between asset prices and sustainable policies is analyzed in Section 5. Section 6 discusses a very simple model of coordination in the short-term nominal bond. Finally, Section 7 concludes.

2 An Overview

In this section I outline the different equilibrium concepts and the main result of the paper. Formal definitions, results, and proofs are in Sections 3 to 5.

2.1 A Simple Economy

I consider an infinitely repeated version of the economy proposed in Barro and Gordon (1983a). Monetary policy is the outcome of a game between two players, the monetary authority and the private sector. At every period $t = 1, 2...$ inflation expectations $\hat{\pi}_t$ are embodied in the private sector's actions, say, by setting nominal prices. The actual inflation rate is $\pi_t \geq \delta > 0$, and is directly set by the monetary authority. A Phillips curve determines aggregate output

$$y_t = y^* + \kappa \left(\pi_t - \hat{\pi}_t\right)$$

where $\kappa > 0$ and $y^* > 0$. Agents value output and inflation at date t according to

$$-\left(\bar{y} - y_t\right)^2 - \left(\pi_t - \delta\right)^2$$

where $\bar{y} > y^*$. The monetary authority is assumed to be benevolent: its objective function is the private-sector welfare. The mismatch between the equilibrium output y^* and the output "target" $\bar{y}$ reflects the presence of distortions.[9] The indirect utility function can be expressed as the negative of the loss function

$$l\left(\pi_t, \hat{\pi}_t\right) = \left(\bar{y} - y^* - \kappa\left(\pi_t - \hat{\pi}_t\right)\right)^2 + \left(\pi_t - \delta\right)^2.$$

The private sector's continuation welfare at any date $t \geq 1$ is

$$v_t = -\sum_{j=t}^{\infty} \delta^j l\left(\pi_j, \hat{\pi}_j\right)$$

where $1 > \delta > 0$ is the intertemporal discount rate.[10]

I start by solving for the optimal monetary policy. Rational expectations imply that there is no room for surprises, i.e., $\hat{\pi} = \pi$. Therefore the optimal monetary policy π^r solves

$$\min_{\pi \geq \delta} l\left(\pi, \pi\right)$$

at all dates. The optimal monetary policy is trivially given by the lower bound, $\pi^r = \delta$. In more nuanced environments the optimal inflation rate may be positive and is usually a function of the state of the economy.

It is well-known that the monetary authority must be able to commit in order to implement the optimal policy in the one-period economy. Say the private sector expects the optimal monetary policy, $\hat{\pi} = \pi^r$, and the monetary authority has a chance to review the policy decision. Since $\hat{\pi}$ is already set, the monetary policy solves the *ex-post* problem:

$$\min_{\pi \geq \delta} l\left(\pi, \hat{\pi}\right).$$

The monetary authority's best response function is

$$\pi = \frac{\kappa\left(\bar{y} - y^*\right) + \delta}{1 + \kappa^2} + \frac{\kappa^2}{1 + \kappa^2}\hat{\pi}$$

[9] See Woodford (2003) for a structural model.

[10] Thus the lower bound on inflation is indeed given by the zero nominal interest rate bound.

and thus the monetary authority would choose to inflate over private-sector expectations if these were based on the optimal monetary policy:

$$\pi = \frac{\kappa\left(\bar{y} - y^*\right) + \delta}{1 + \kappa^2} + \frac{\kappa^2}{1 + \kappa^2}\pi^r > \pi^r.$$

Hence, the optimal monetary policy is not compatible with rational expectations and policy discretion.

What then is the rational expectations outcome in the one-period economy without commitment? We are looking for a fixed point in the best response function. That is, the monetary authority must find it optimal to validate the private-sector expectations,

$$\hat{\pi} = \arg\min_{\pi \geq \delta} l\left(\pi, \hat{\pi}\right).$$

It is straightforward to solve for the unique one-period Nash equilibrium π^n,

$$\pi^n = \delta + \kappa\left(\bar{y} - y^*\right).$$

Hence, the lack of commitment leads to the well-known inflation bias, $\pi^n > \pi^r$.

2.2 Sustainable Policy

In the infinite horizon economy, reputation can substitute for commitment and lower inflation. Say the private sector initially expects inflation to be below the one-period Nash equilibrium, $\hat{\pi} < \pi^n$. If monetary policy deviates at any date from expectations, $\pi_t \neq \hat{\pi}$, then the private sector subsequently coordinates on the one-period Nash equilibrium π^n. This strategy "punishes" the monetary authority whenever it exploits the Phillips curve. If the threat of higher inflation in future periods is large enough to deter the monetary authority, then it is possible to sustain lower inflation rates than π^n.

Whether any inflation path $\{\pi_t\}_{t=1}^{\infty}$ is *sustainable* in this manner is easy to check. At every date $t \geq 1$, the monetary authority can fulfill private-sector expectations and get the continuation value associated with $\{\pi_j\}_{j=t}^{\infty}$,

$$v_t = -\sum_{j=t}^{\infty} \delta^j l\left(\pi_j, \pi_j\right)$$

or it can choose some $\pi' \neq \pi_t$, knowing the one-period Nash equilibrium will follow forever after

$$-l\left(\pi', \pi_t\right) - \frac{\delta}{1 - \delta} l\left(\pi^n, \pi^n\right).$$

The inflation path $\{\pi_t\}_{t=1}^{\infty}$ is a *sustainable plan* if the monetary authority chooses to fulfill private-sector expectations,

$$v_t \geq -l\left(\pi', \pi_t\right) - \frac{\delta}{1-\delta} l\left(\pi^n, \pi^n\right) \tag{1}$$

for all $\pi' \geq \delta$ at all dates $t \geq 1$.[11] In this simple economy, a sustainable inflation path can also be thought of as a *sustainable policy.*

The Folk theorem says that if the intertemporal discount rate is low enough, i.e., δ close enough to one, the optimal monetary policy should be sustainable. It is easy to check this. The continuation value associated with the infinite repetition of π^r is

$$v^r = -\frac{\left(\bar{y} - y^*\right)^2}{1-\delta}$$

while the best deviation achieves

$$-\frac{\left(\bar{y} - y^*\right)^2}{1+\kappa^2} - \frac{\delta}{1-\delta}\left(1+\kappa^2\right)\left(\bar{y} - y^*\right)^2$$

so the optimal monetary policy will be sustainable if

$$\frac{1}{1-\delta} \leq \frac{1}{1+\kappa^2} + \frac{\delta}{1-\delta}\left(1+\kappa^2\right)$$

or

$$\delta \geq \frac{1}{2+\kappa^2}.$$

Unfortunately, there are a lot of sustainable inflation paths. To start with, the infinite repetition of the one-period Nash equilibrium, $\pi_t = \pi^n$ for all $t \geq 1$, is always a sustainable plan. It can be shown that, if the optimal monetary policy is sustainable, then any constant sequence in $[\pi^r, \pi^n]$ is sustainable too. Actually, the equilibrium multiplicity does not stop at deterministic sequences: it is also possible to sustain stochastic processes for inflation.

I will be interested in the *best sustainable policy or plan* $\{\pi_t^*\}_{t=1}^{\infty}$, i.e., the sustainable inflation path that achieves the highest value at date $t = 1$, v_1. The optimal monetary policy, $\pi_t = \pi^r$ for all $t \geq 1$, may be a sustainable plan, in which case it trivially is the best sustainable plan. The cases of interest are those in which the optimal monetary policy is not sustainable, yet the best sustainable plan improves upon the one-period Nash equilibrium.

[11]The concept of a sustainable plan used in the later sections is broader: reversion to Nash equilibrium is not the only punishment possible. Condition (1) is sufficient but not necessary to establish an inflation path as sustainable.

The results of this paper are built upon three key properties of the best sustainable plan. First, it is a constant sequence, $\pi_t^* = \pi^*$.[12] Second, it achieves the *lowest* inflation rate sustainable at every date. Third, as long as the lower bound on inflation is not binding, the only sustainable inflation path with $\pi_1 = \pi^*$ is the best sustainable plan.

In the main text I work on the theory developed by Abreu et al. (1990) to formally prove these properties. Here I just sketch the logic behind them. Let $\{v_t^*\}_{t=1}^{\infty}$ be the sequence of continuation values associated with the best sustainable inflation path $\{\pi_t^*\}_{t=1}^{\infty}$. I can write the sustainability condition (1) at $t = 1$ as

$$v_1^* = -l\left(\pi_1^*, \pi_1^*\right) + \delta v_2^* \geq -l\left(\pi', \pi_1^*\right) - \frac{\delta}{1-\delta} l\left(\pi^n, \pi^n\right)$$

for all $\pi' \geq \delta$. Why is the best sustainable plan a constant sequence? If the continuation value were decreasing $v_1^* > v_2^*$, then one can just "postpone" the start of the best sustainable plan and do better. Note the alternative sequence $\{\pi_1^*, \pi_1^*, \pi_2^*, ...\}$ is sustainable since

$$-l\left(\pi_1^*, \pi_1^*\right) + \delta v_1^* > v_1^* \geq -l\left(\pi', \pi_1^*\right) - \frac{\delta}{1-\delta} l\left(\pi^n, \pi^n\right)$$

and it achieves strictly more welfare than $\{\pi_t^*\}_{t=1}^{\infty}$. A similar argument can be made if the continuation value were increasing $v_1^* < v_2^*$ by considering an alternative sequence $\{\pi_2^*, \pi_3^*, ...\}$ starting at date $t = 1$. Therefore the best sustainable plan achieves a constant value $v_t^* = v^*$ for all $t \geq 1$, which is only possible with a constant inflation rate $\pi_t^* = \pi^*$.

The second property follows easily now. Say a sustainable inflation path $\{\tilde{\pi}_t\}_{t=1}^{\infty}$ is such that, at some date t, $\tilde{\pi}_t < \pi^*$. The sustainability condition (1) at date t implies

$$-l\left(\tilde{\pi}_t, \tilde{\pi}_t\right) + \delta \tilde{v}_{t+1} \geq -l\left(\pi', \tilde{\pi}_t\right) - \frac{\delta}{1-\delta} l\left(\pi^n, \pi^n\right)$$

for all $\pi' \geq \delta$. Since $v^* \geq \tilde{v}_{t+1}$, the inflation path $\{\tilde{\pi}_t, \pi^*, \pi^* ...\}$ is also sustainable. Given that $l(\pi, \pi)$ is increasing in π, $\{\tilde{\pi}_t, \pi^*, \pi^* ...\}$ achieves

$$-l\left(\tilde{\pi}_t, \tilde{\pi}_t\right) + \delta v^* > v^*$$

which contradicts the definition of a best sustainable plan.

Finally, the proof of the third property also proceeds by contradiction. Assume a sequence $\{\pi^*, \{\tilde{\pi}_t\}_{t=2}^{\infty}\}$ is sustainable, with $\tilde{\pi}_t \neq \pi^* > \delta$ at some date $t \geq 2$. Condition (1) implies

$$-l\left(\pi^*, \pi^*\right) + \delta \tilde{v}_2 \geq -l\left(\pi', \pi^*\right) - \frac{\delta}{1-\delta} l\left(\pi^n, \pi^n\right).$$

[12]This generalizes to the best sustainable plan being a function of the exogenous state of the economy.

Since I have already shown that the best sustainable plan is $\pi_t = \pi^*$, the continuation sequence $\{\tilde{\pi}_t\}_{t=2}^{\infty}$ achieves strictly less $\tilde{v}_2 < v^* = -\frac{l(\pi^*,\pi^*)}{1-\delta}$. Hence

$$-\frac{l(\pi^*,\pi^*)}{1-\delta} > -l(\pi',\pi^*) - \frac{\delta}{1-\delta} l(\pi^n,\pi^n)$$

for all $\pi' \geq \delta$. Since the loss function l is continuous in both arguments, there exists $\hat{\pi} = \pi^* - \varepsilon$ for arbitrarily small $\varepsilon > 0$ such that

$$-\frac{l(\hat{\pi},\hat{\pi})}{1-\delta} \geq -l(\pi',\hat{\pi}) - \frac{\delta}{1-\delta} l(\pi^n,\pi^n).$$

Hence the constant inflation path $\pi_t = \hat{\pi}$ is sustainable. Since $l(\hat{\pi},\hat{\pi}) < l(\pi^*,\pi^*)$, $\pi_t = \hat{\pi}$ for all $t \geq 1$ achieves more welfare than the best sustainable plan $\hat{v} > v^*$, a contradiction.[13]

2.3 Pricing Nominal Assets

To make the point of this paper clear, I introduce asset markets in a way that neither the agent's payoff nor action sets are modified by asset trading. As a result, the set of sustainable inflation paths is left intact and asset prices are confined to a coordination role.

All asset trade happens at date $t = 0$, one period before the game as described above starts. For simplicity, only a one-period nominal bond is traded here, sold at discount rate Q. There is no monetary phenomena at date $t = 0$. All private-sector agents get an exogenous endowment $y_0 < \bar{y}$.

Private-sector agents optimally choose their bond holdings $\hat{B}$. Abstracting from borrowing constraints, the no-arbitrage condition implies

$$Q = \delta \frac{(\bar{y} - c_1)}{(\bar{y} - c_0)} \frac{P_0}{P_1}.$$

I assume that the monetary authority holdings B are exogenously dictated by a fiscal authority. Lump-sum transfers balance the monetary authority budget constraint and thus the resource constraint equates consumption with output. This renders the level of bond holdings irrelevant for the monetary authority decision problem at any date. It also does not affect the private-sector action set. As intended, the assumptions ensure that the set of sustainable plans is preserved.

[13] Readers familiar with Abreu et al. (1990) will recognize the bang-bang property of the best sequential equilibrium.

An asset market equilibrium combines a sustainable inflation path, the portfolio decision and the market clearing conditions. Since there are no surprises along the equilibrium path, the arbitrage condition simplifies to a simple Fischer equation,

$$Q = \delta \frac{\bar{y} - y^*}{\bar{y} - y_0} \frac{1}{\pi_1}. \tag{2}$$

I assume that there is no sustainable inflation path that makes the zero nominal interest rate condition $Q \leq 1$ bind.

The arbitrage condition (2) is a one-to-one mapping between the nominal interest rate and inflation in the first period. However, the nominal interest rate does not seem to be informative with respect to the inflation rate in later periods. Moreover, not even the one-to-one relationship with period $t = 1$ inflation stands once the possibility of sustainable stochastic inflation paths is taken into account. There are usually many sustainable inflation paths compatible with the same nominal interest rate at date $t = 0$.

The key result of this paper is that, despite the limited scope of the arbitrage condition (2), there is a robust one-to-one relationship with the best sustainable plan. That is, if the asset market clears at price

$$Q^* = \delta \frac{\bar{y} - y^*}{\bar{y} - y_0} \frac{1}{\pi^*}$$

the only compatible inflation path is the best one, $\pi_t = \pi^*$ for all $t \geq 1$.

The properties of the best sustainable plan discussed in the previous subsection are now instrumental. First and foremost, it is sufficient to pin down the date $t = 1$ inflation rate at π^* to ensure that the best sustainable plan follows. Second, there is no room for stochastic inflation paths because π^* is the lowest sustainable inflation rate. Sitting at one extreme of the support for sustainable inflation rates, only degenerate lotteries are compatible with discount rate Q^*. It is also straightforward that Q^* is the highest discount price among the set of asset market equilibria.

The nominal interest rate also coordinates expectations into the best sustainable equilibrium in alternative market structures. In Section 4 I formally prove the result for a richer set of assets in a stochastic environment. Yet the result still hinges exclusively on the short-term nominal rate.[14]

However, a more realistic modeling of the asset market is bound to modify the set of sustainable plans. Nominal bond holdings are likely to shape the ex-post incentives of the monetary authority—see, for example, Calvo (1978). Moreover, the short-term interest rate is often not only a nominal anchor but also the operative instrument of monetary policy. I abstracted from these possibilities in order to focus on the coordination role.

[14]It is also possible, but cumbersome, to prove the result in a sequence-of-markets equilibrium.

3 Sustainable Monetary Policy

This is an infinite horizon economy $t = 1, 2, ...$ Monetary policy is characterized as the outcome of a repeated game between the monetary authority and a continuum I of agents, which are denoted as a whole as the private sector.

At the beginning of each period $t \geq 1$, the exogenous state s_t becomes common knowledge. I assume that the exogenous state $s_t \in S$ is governed by a stochastic first-order Markov probability process μ.

After the exogenous state is observed, private-sector agents choose some variable, $z_{it} \in Z$. For example, some firms may set nominal prices in advance or unions may agree on nominal wages for the period. Let $z_t = \int z_{it} di$ be the average of all agents decisions. I will assume that the private-sector agent's and monetary authority's payoffs depend on actions $\{z_{it}\}_I$ only through its average z_t. This implies that a private-sector competitive equilibrium can be thought of as the average action z_t^* solving the optimality conditions $z_t^* = \zeta(z_t^*, \hat{\pi}_t)$, where $\hat{\pi}_t$ is the private-sector inflation expectation. Furthermore, I assume that, given an inflation expectation $\hat{\pi}_t$, the private-sector competitive equilibrium is unique. Hence, the private-sector competitive equilibrium can be summarized by $\hat{\pi}_t$, which is treated as the "action" of the private sector.

Finally, the monetary authority sets the actual inflation rate π_t taking as given the private sector's inflation expectations $\hat{\pi}_t$. Inflation is bounded below by feasibility, $\pi_t \in \Pi \equiv \{\pi \geq \delta\}$ where $\delta > 0$.

The economy considered is very simple. Output and inflation are linked by a classic Phillips curve,

$$y_t = y(\pi_t - \hat{\pi}_t; s_t)$$

where $y(\pi_t - \hat{\pi}; s_t)$ is strictly positive, increasing in the first argument, and bounded above. Let $y^*(s_t) \equiv y(0, s_t)$. Output is transformed one-to-one into a non-storable consumption good. The aggregate resource constraint is then $c_t \leq y_t$.

The household period payoff is given by $u(y_t) - g(\pi_t)$, where u is a strictly increasing, concave, and twice differentiable function; and g is a strictly increasing and twice differentiable equation. Function u is assumed to be bounded below and g bounded above. Finally, I assume that the monetary authority is benevolent so its period payoff is given by $u(y_t) - g(\pi_t)$ as well.[15]

I briefly introduce the time-inconsistency problem in a one-period version of this economy. First, I will define an one-period Nash equilibrium—the discretionary equilibrium in the

[15]Foundations for this economy are discussed in Woodford (2003) and references. The use of a classical Phillips curve formulation is very convenient. The no-reputation equilibrium is basically an infinite sequence of static economies—hence the only intertemporal link possible is reputation. For an extension of infinitely repeated monetary policy games with endogenous state variables, see Chang (1998).

language of Barro and Gordon (1983b). I will also characterize the one-period optimal monetary policy.

Definition 1 *A one-period Nash equilibrium at state s is $\pi \geq \delta$ such that*

$$u\left(y^*\left(s\right)\right) - g\left(\pi\right) \geq u\left(y\left(\pi' - \pi; s\right)\right) - g\left(\pi'\right)$$

for all $\pi' \geq \delta$.

I will assume that there exists a one-period Nash equilibrium for each state $s \in S$.

Definition 2 *The optimal monetary policy at state s is $\pi \geq \delta$ such that*

$$u\left(y^*\left(s\right)\right) - g\left(\pi\right) \geq u\left(y^*\left(s\right)\right) - g\left(\pi'\right)$$

for all $\pi' \geq \delta$.

The optimal monetary policy is trivially given by $\pi = \delta$ for all $s \in S$.[16] The time-inconsistency problem arises when the two equilibrium concepts are not equivalent, i.e., the optimal monetary policy is not the unique one-period Nash equilibrium. If this is the case, some commitment technology is required to implement the optimal monetary policy as the unique rational expectations equilibrium.

The set of Nash equilibrium is usually enlarged in the infinite horizon economy. In particular, history-dependent equilibria arise, and some of the equilibria feature trigger strategies that resemble reputation.

I start by introducing the notation for histories.[17] Let $\pi^t = \{\pi_1, \pi_2..., \pi_t\} \in \Pi^t$ denote the history of the monetary authority's actions up to date $t \geq 1$. In a similar fashion define $\hat{\pi}^t \in \Pi^t$ and $s^t \in S^t$ for $t \geq 1$. To denote the set of continuation histories at s^t, I use $S^j|s^t$, $j \geq t$. The payoff-relevant node is $h^t = \{\pi^t, \hat{\pi}^t, s^t\} \in H^t \equiv \Pi^{2t} \times S^t$ for $t \geq 1$, with $h^0 = \{\varnothing\}$. Node h^t is also the end-of-the-period history.

A consumption plan $c = \{c\left(h^t\right) : h^t \in H^t, t \geq 1\}$ specifies a level of consumption at every node $h^t \in H^t$, $t \geq 1$. Let F be a probability measure over all game nodes H^t, $t \geq 1$. The private-sector welfare at any h^t, $t \geq 1$, is defined over c given F

$$v\left(c; h^t, F\right) = \sum_{j=0}^{\infty} \sum_{s^{t+j} \in S^{t+j}} \delta^j \mu\left(s^{t+j}|s^t\right) \int_{H^{t+j}} \left(u\left(c\left(h^{t+j}\right)\right) - g\left(\pi_{t+j}\right)\right) F\left(dh^{t+j}|h^t, s^{t+j}\right) \quad (3)$$

[16] This "boring" optimal monetary policy ensures that the analysis is focused on how reputation can achieve *lower* average inflation.

[17] I assume that strategies can only be conditional on public histories and not on the history of private actions $\{z_{it}\}_{t=1}^{\infty}$.

where $0 < \delta < 1$ is the intertemporal discount rate. I stretch the definition of v to ex-ante welfare

$$v(c;s,F) = \int_H v\left(c;h^1,F\right) F\left(dh^1|s\right), \tag{4}$$

$$v(c;F) = \sum_{s\in S} \mu(s|s_0)\, v(c;s,F). \tag{5}$$

Finally, the resource constraint is given by

$$c\left(h^t\right) = y\left(\pi_t - \hat{\pi}_t; s_t\right) \tag{6}$$

for all $h^t \in H^t$, $t \geq 1$.[18]

Now I describe the strategy space. For the private sector, the relevant decision node is (h^{t-1}, s_t) as the present state of the economy is known in addition to the history of past actions.[19] The period strategy is denoted by $\hat{\sigma}(h^{t-1}, s_t) \in \Delta(\Pi)$. The strategy space contains mixed strategies in order to allow equilibria where private-sector expectations are driven by a sunspot variable. The private-sector strategy is then

$$\hat{\sigma} = \left\{\hat{\sigma}\left(h^{t-1}, s_t\right) : \left(h^{t-1}, s_t\right) \in H^{t-1} \times S^t, t \geq 1\right\}.$$

The monetary authority is aware of the history of actions, the state of the economy, and the realization of the private-sector strategy. The relevant node for the monetary authority is then $x^t \equiv (h^{t-1}, s_t, \hat{\pi}_t) \in X^t \equiv H^{t-1} \times S^t \times \Pi$ and its period strategy is $\sigma(x^t) \in \Pi$. Let

$$\sigma = \left\{\sigma\left(x^t\right) : x^t \in X^t, t \geq 1\right\}$$

denote the monetary authority strategy.

The chosen equilibrium concept is the sequential equilibrium.

Definition 3 *A sequential equilibrium (SE) is a strategy profile $\{\hat{\sigma}, \sigma\}$, a probability distribution F and a consumption plan c such that:*

1. *For all $x^t \in X^t, t \geq 1$*

$$v\left(c;\left\{x^t, \sigma\left(x^t\right)\right\}, F\right) \geq v\left(c;\left\{x^t, \pi'\right\}, F\right)$$

for all $\pi' \geq \delta$;

[18] The resource constraint is stated with strict equality sign. This turns out to be important since it rules out punishment strategies based on disposal of resources—what are known as "burning money" strategies.

[19] It may sound awkward to talk of the private-sector *strategy* given that the private-sector is not a rational player but a continuum of small agents. The distinction will be contained in the equilibrium definition; the term strategy is just convenient.

2. *For all* $(h^{t-1}, s_t) \in H^{t-1} \times S, t \geq 1$*,* $\hat{\sigma}$ *is such that*

$$\hat{\pi}_t = \sigma\left(x^t\right)$$

almost everywhere in $H^t|X^t$*;*

3. *For all* $h^t \in H^t, t \geq 1$*, the resource constraint (6) holds;*

4. *Probability measure* F *is generated from* $\{\hat{\sigma}, \sigma\}$*.*

The key elements of the sequential equilibrium definition are the subgame perfection requirement and rational expectations. The former ensures that the monetary authority's decision maximizes private-sector welfare at every decision node—this is Condition 1 in the definition. Condition 2 imposes rational expectations: it implies that the private sector has the correct beliefs with respect to the actual inflation rate along the equilibrium path. Note that Conditions 1 and 2 are very different. The private-sector "player," actually a continuum of atomistic agents, does not behave strategically.

It is straightforward to show that the infinitely repeated version of a one-period Nash equilibrium is a SE. Usually there will be many SE with distinct welfare properties. The SEs that deliver the highest welfare possible are of obvious interest.

Definition 4 *A best sequential equilibrium (BSE) is a sequential equilibrium* $\{\sigma, \hat{\sigma}, F, c\}$ *such that*

$$v\left(c; F\right) \geq v\left(c'; F'\right)$$

for all sequential equilibria $\{\sigma', \hat{\sigma}', F', c'\}$*.*

Before proceeding to the discussion of the equilibrium properties, I define a monetary policy plan as a probability measure M over $\Pi^t \times S^t$. Clearly, for a given SE $\{\sigma, \hat{\sigma}, F, c\}$, the probability measure F defines a monetary policy plan. Following Chari and Kehoe (1990), a monetary policy plan will be sustainable if there exists a SE generating the policy plan.

Definition 5 *A probability measure* M *over* $\Pi^t \times S^t$ *is a sustainable monetary policy plan if there exists a sequential equilibrium* $\{\sigma, \hat{\sigma}, F, c\}$ *such that* M *is generated by* F*.*

I will often use sustainable plan for short.

3.1 Sustainable Equilibrium Properties

A key object in the analysis of SE is the equilibrium value set, i.e., the set of private-sector welfare associated with a SE. It will prove instrumental to characterizing the BSE. For convenience, I first define the equilibrium value set conditional on the state s,

$$V(s) \equiv \{v(c; s, F) \,|\, \{\sigma, \sigma', F, c\} \text{ is a SE}\}$$

and then the ex-ante welfare set

$$V \equiv \{v(c; F) \,|\, \{\sigma, \sigma', F, c\} \text{ is a SE}\}.$$

As mentioned earlier, the infinitely repeated version of a one-period Nash equilibrium is a SE. Hence $V(s)$ is not empty for all $s \in S$.

The remainder of this section builds on the seminal work of Abreu et al. (1990) to characterize the set $V(s)$ and then the BSE. The private-sector welfare can be written recursively. From (3), the end of period welfare can be decomposed in the period payoff and the continuation value,

$$v\left(c; h^t, F\right) = u\left(c\left(h^t\right)\right) - g(\pi_t) + \delta \sum_{s^{t+1}} \mu\left(s^{t+1}|s^t\right) \int_{H^{t+1}} v\left(c; h^{t+1}, F\right) F\left(dh^{t+1}|h^t\right).$$

The key observation is that every subgame in a SE constitutes a SE as well. This implies that, at any node of the game, the continuation payoffs must belong to $V(s)$ as well, i.e.,

$$\int_{H^t} v\left(c; h^t, F\right) F\left(dh^t|h^{t-1}, s_t\right) \in V(s_t)$$

for all $(h^{t-1}, s_t) \in H^{t-1} \times S$, $t \geq 1$.

The next step is to define a sustainable action $\pi \in \Pi$. Sustainable actions are the building blocks of a SE. Loosely speaking, for an action π_t to be sustainable, there should be subgame-perfect strategies such that the monetary authority validates the private-sector expectations $\hat{\pi}_t = \pi_t$. These strategies must specify a plan of action for every possible monetary authority action, and each of these plans must be sustainable.

To characterize an action as sustainable seems potentially complicated. It is possible, though, to easily characterize sustainable actions in terms of continuation values: for an action π to be sustainable, there should be a schedule of continuation values in $\{V(s)\}_{s \in S}$ such that the monetary authority validates the private-sector expectations $\hat{\pi} = \pi$.

Definition 6 *An action* $\pi \in \Pi$ *is a sustainable action at* $s \in S$ *if there exists vectors* $w_1, w_2 \in \{V(s')\}_{s' \in S}$ *such that*

$$\begin{aligned} & u(y^*(s)) - g(\pi) + \delta \sum_{s'} \mu(s'|s) w_1(s') \\ \geq \; & u(y(\pi' - \pi; s)) - g(\pi') + \delta \sum_{s'} \mu(s'|s) w_2(s') \end{aligned}$$

for all $\pi' \geq \delta$.

Vector w_1 details the continuation value for each state s' if the monetary authority validates the private-sector expectations; loosely speaking, it is the "reward." The "punishment" vector w_2 states the continuation value if the monetary authority does not validate the private-sector expectations. Because continuation values belong to $\{V(s)\}_{s \in S}$, they correspond to some SE so there exist sub-game perfect strategies sustaining the action π. Note also the sustainable actions are defined with a simple punishment rule: any deviation π' has the same punishment, i.e., the same continuation value vector w_2. These simple punishment rules are without loss of generality, as first pointed out in Abreu (1988).

The next theorem states the key results from Abreu et al. (1990) used in this paper. Hopefully the previous discussion has introduced the reader to the recursive structure of SE and the role of sustainable actions.

Theorem 1 *For all* $s \in S$, $V(s)$ *is a non-empty and compact set in* $\Re$. *Moreover, for any sequential equilibrium* $\{\sigma, \hat{\sigma}, F, c\}$ *and for all nodes* $h^{t-1} \in H^{t-1}$, $t \geq 1$,

1. *The private-sector welfare satisfies*

$$v\left(c; h^j, F\right) \in V(s)$$

   almost everywhere in $H^j|h^{t-1}$, $j \geq t-1$;

2. *Strategy* σ *specifies sustainable actions almost everywhere* $X^j|h^{t-1}$, $j \geq t-1$.

Finally, for any profile of sustainable actions $\{\pi(s)\}_{s \in S}$ *and a date* $t \geq 1$, *there exists a sequential equilibrium such that* F *assigns probability one to* $\{\pi(s)\}_{s \in S}$ *at date* t.

Proof. See Abreu et al. (1990). For a simpler proof restricted to perfect monitoring games such as the present one, see Cronshaw and Luenberger (1994) or Ljungqvist and Sargent (2000) ■

It is useful to rephrase Theorem 1 above in terms of sustainable plans. First, it implies that a sustainable plan M assigns positive probability only to sustainable actions. Moreover, for a given profile of sustainable actions and a given date, there exists at least one sustainable plan implementing these actions at the given date with probability 1. However, Theorem 1 does not imply that any probability measure M over sustainable actions is sustainable.

Everything is in place to characterize the BSE and the most important result of this section. Using the results in Theorem 1, I show that the BSE features the lowest sustainable inflation rate along the whole equilibrium path. Hence, while strategies are history dependent in a potentially complicated way, the best sustainable plan M^* is unique and strikingly simple: along the equilibrium path inflation depends only on the current state s.

Proposition 1 *A sequential equilibrium is a best sequential equilibrium if and only if $\sigma(x^t) = \pi^*(s_t)$ almost everywhere in X^t for all $t \geq 1$ where*

$$\pi^*(s) = \inf\{\pi | \pi \text{ is sustainable at } s\}.$$

Proof. Using the fact that $V(s)$ is compact, π is sustainable if and only if

$$\delta \sum_{s'} \mu(s'|s)(\sup\{V(s')\} - \inf\{V(s')\}) \geq u(y(\pi' - \pi; s)) - u(y^*(s)) - (g(\pi') - g(\pi)) \quad (7)$$

for all $\pi' \geq \delta$.

I will use this to prove that the set of sustainable actions is non-empty and compact for all $s \in S$. First, the set always includes any one-period Nash equilibrium and (7) follows trivially. Consider a convergent sequence $\{\pi_j\}_{j=0}^{\infty} \to \tilde{\pi}$ of sustainable actions at s. Since the right-hand side of 7 is a measurable function for all $\pi' \geq \delta$, it follows that $\tilde{\pi}$ satisfies (7). Hence $\pi^*(s)$ exists and it is a sustainable action at s.

The second part of the proof shows that $\pi^*(s)$ effectively characterizes the BSE. First, I prove that $V(s)$ is bounded above by

$$\bar{v}(s) = u(y(\pi^*(s))) - g(\pi^*(s)) + \delta \sum_{s'} \mu(s'|s_1) \sup\{V(s')\}.$$

To prove this, assume there exists a SE with strictly larger value $v\left(\tilde{c}; s_1, \tilde{F}\right) > \bar{v}(s_1)$ for $s_1 \in S$. Because all actions with positive probability in a SE are sustainable actions, $v\left(\tilde{c}; h^2, \tilde{F}\right) \leq \sup\{V(s_2)\}$ almost everywhere in H^2. Hence $v\left(\tilde{c}; s_1, \tilde{F}\right) > \bar{v}(s_1)$ implies that with positive probability

$$u(y(\pi^*(s_1))) - g(\pi^*(s_1)) < u(y(\pi^*(s_1))) - g\left(\tilde{\pi}\left(h^1\right)\right)$$

and thus $g(\tilde{\pi}(h^1)) < g(\pi^*(s_1))$. But $g(\pi)$ is strictly increasing in π. Therefore it would imply that there exists a sustainable inflation rate $\tilde{\pi}(h_1)$ strictly lower than $\pi^*(s_1)$, which violates the definition of $\pi^*(s_1)$.

It follows that $\bar{v}(s_1) = \sup\{V(s_1)\}$—from Theorem 1 $\bar{v}(s_1) \in V(s_1)$ as there exists a SE featuring $\pi^*(s_1)$ at date $t = 1$.

The same construct for any $s \in S$ implies

$$\sup\{V(s)\} = u(y^*(s)) - g(\pi^*(s)) + \delta \sum_{s'} \mu(s'|s) \sup\{V(s')\} \tag{8}$$

for all $s \in S$. The system of equations described by (8) proves the if and only if part of the proposition. ■

Next I present a simple corollary of Proposition 1 that plays an important part in the result. It implies that the BSE must be sustained by promising the highest reward and threatening with the upmost punishment. Hence, if at date $t = 1$ the best sustainable inflation $\pi^*(s_1)$ is achieved for all realizations of s_1, it must be that the BSE follows. The corollary highlights the importance of the recursive structure of the SE.[20]

Corollary 1 *If $\pi^*(s) > \delta$, a sequential equilibrium with $\sigma(x^1) = \pi^*(s_1)$ almost everywhere in X^1 is a best sequential equilibrium.*

Proof. Assume there exists a SE $\left\{\tilde{\sigma}, \tilde{\sigma}', \tilde{c}, \tilde{F}\right\}$ with $\tilde{\sigma}(x^1) = \pi^*(s_1)$ a.e. in X^1 but $v\left(\tilde{c}; s_1, \tilde{F}\right) < \sup\{V(s_1)\}$. It then implies that there exists a reward continuation value $w_1$which renders $\pi^*(s_1)$ sustainable with $\delta \sum_{s'} \mu(s'|s_1)(\sup\{V(s')\} - w_1(s')) > 0$. Since $\pi^*(s_1) > \delta$, then there exists $\pi^{**} < \pi^*(s_1)$ satisfying (7) and contradicting the definition of $\pi^*(s_1)$ ■

4 Date $t = 0$ Asset Markets

In this section, I introduce trade in nominal and real assets. I want to emphasize the contribution of this paper, which is to point out the coordination role of asset prices. In order to avoid any confusion, I lay out a trade structure that provides asset prices but preserves the monetary policy game exactly as described above. Neither payoffs nor action sets are modified; in a precise sense to be made clear later, the equilibrium value set is left intact by asset trade.

[20]To be precise, it is an application of the bang-bang result discussed in Abreu et al. (1990).

4.1 Asset Market Environment and Equilibrium Definition

In addition to the economy described above, I introduce a $t = 0$ period when all asset trading takes place. At date $t = 0$, the private sector has an endowment $y_0 > 0$ and there are no monetary decisions.

There are real and nominal assets that are simultaneously traded at date $t = 0$. Available real assets are a full set of contingent claims $b(s^t)$, sold at discount price $q(s^t)$, which deliver one unit of output at node $s^t \in S^t$, $t \geq 1$. There is also a set of nominal claims $B(s^t)$, sold at discount price $Q(s^t)$, which deliver one nominal unit in the event $s^t \in S^t, t \geq 1$. All asset prices are in units of date $t = 0$ consumption. Repayment of all claims is perfectly enforceable. Let $q = \{q(s^t) : s^t \in S^t, t \geq 1\}$ and similarly for Q, b and B.

It is important to note that asset markets are not complete. The history of actions $h^t \in H^t$ introduces income volatility. Yet assets only span exogenous histories S^t. When I discuss the results, it will be clear that the results go through with a more limited asset structure or sequential trading.

The price level $P(h^t)$ at node h^t is given by

$$P\left(h^t\right) = \pi_t P\left(h^{t-1}\right) \tag{9}$$

for all $h^t \in H^t$, $t \geq 1$. As the price level is irrelevant at $t = 0$, $P(h^0) = 1$ solves the nominal indeterminacy. Let $P = \{P(h^t) : h^t \in H^t, t \geq 0\}$.

The monetary authority's net asset holdings, denoted $\{b, B\}$, are exogenously dictated by the government. To avoid any interaction between fiscal and monetary policy, I assume that the government budget is cleared via lump-sum taxes $\tau = \{\tau(h^t) : h^t \in H^t, t \geq 0\}$. Therefore, the transfers satisfy

$$\sum_{t=1}^{\infty} \sum_{s^t \in S^t} \left(q\left(s^t\right) b\left(s^t\right) + Q\left(s^t\right) B\left(s^t\right)\right) \leq \tau_0 \tag{10}$$

and

$$0 \leq b\left(s^t\right) + \frac{B(s^t)}{P(h^t)} + \tau\left(h^t\right) \tag{11}$$

for all $h^t \in H^t$, $t \geq 1$.

The private-sector date $t = 0$ asset market problem is to choose the date $t = 0$ consumption c_0, a consumption plan $c = \{c(h^t) : h^t \in H^t, t \geq 1\}$, and real and nominal assets $\left\{\hat{b}, \hat{B}\right\}$ to solve

$$\max u(c_0) + \delta \sum_{s_1 \in S_1} \mu(s_1|s_0)\, v(c; s_1, F) \tag{12}$$

subject to

$$c_0 + \sum_{t=1}^{\infty} \sum_{s^t \in S^t} \left(q\left(s^t\right) \hat{b}\left(s^t\right) + Q\left(s^t\right) \hat{B}\left(s^t\right) \right) \leq y_0 - \tau_0, \tag{13}$$

and for all $h^t \in H^t$, $t \geq 1$,

$$c\left(h^t\right) \leq \hat{b}\left(s^t\right) + \frac{\hat{B}\left(s^t\right)}{P\left(h^t\right)} + y\left(\pi_t, \hat{\pi}_t; s_t\right) - \tau\left(h^t\right) \tag{14}$$

as well as non-negative consumption $c\left(h^t\right) \geq 0$ and borrowing constraints

$$\begin{aligned} \hat{b}\left(s^t\right) &\geq -\bar{b}, \\ \hat{B}\left(s^t\right) &\geq -\bar{B}, \end{aligned}$$

for all $s^t \in S^t$, $t \geq 1$ with $\bar{b} > 0$ and $\bar{B} > 0$. Importantly, the private sector takes as given all asset prices, the monetary authority's actions as well as the lump sum taxes τ.

I will assume that all bonds are in zero aggregate supply. Asset market clearing then implies that for all $s^t \in S^t$, $t \geq 1$,

$$\begin{aligned} B\left(s^t\right) + \hat{B}\left(s^t\right) &= 0, \\ b\left(s^t\right) + \hat{b}\left(s^t\right) &= 0. \end{aligned}$$

I also assume that the monetary net asset holdings $\{b, B\}$ never bind the private-sector borrowing constraints.

A date $t = 0$ asset market equilibrium combines a SE, exactly as defined above, with optimal portfolio and market clearing conditions. The key output is the nominal asset prices.

Definition 7 *A date $t = 0$ asset market equilibrium (AME) consists of asset prices $\{q, Q\}$, a price level path P, lump-sum taxes τ, asset holdings $\left\{b, \hat{b}, B, \hat{B}\right\}$, date $t = 0$ consumption c_0, and a sequential equilibrium $\{\sigma, \hat{\sigma}, F, c\}$ such that*

1. *assets $\left\{\hat{b}, \hat{B}\right\}$ and consumption decisions $\{c_0, c\}$ solve the private-sector date $t = 0$ asset market problem, (12);*
2. *date $t = 0$ consumption is feasible $c_0 \leq y_0$,*
3. *prices $\{q, Q\}$ clear all asset markets,*
4. *lump-sum taxes satisfy (10) and (11),*

5. the price level is given by (9) for all $h^t \in H^t$, $t \geq 0$.

I have not considered the possibility that the private sector holds monetary balances, so the nominal interest rate could be negative. To avoid this, I assume that the lowest sustainable inflation $\pi^*(s)$ and the real interest rate are non-negative.

Condition 1 *For all $s \in S$,*

$$\pi^*(s) \geq 1 \geq \delta \frac{u^c(y^*(s))}{u^c(y_0)}.$$

5 Asset Prices and the Best Sustainable Equilibrium

In this section I analyze the mapping between two components of a date $t = 0$ asset market equilibrium: the asset prices and the sustainable equilibrium. This mapping is found to be a correspondence, but the key result is actually characterized by an exception: the relationship between the short-term nominal interest rate and the best sustainable equilibrium is one-to-one.

I start by proving the claim that asset trading has not changed the equilibrium value set. Of course, I disregard the trivial difference induced by the inclusion of period $t = 0$. I define the equilibrium value set for date $t = 0$ asset market equilibrium (AME) as

$$\tilde{V} = \{v(c; F) \mid \{F, c\} \text{ are part of a AME}\}.$$

By definition, an AME contains a SE; thus $\tilde{V} \subset V$. Proposition 2 proves that $V \subset \tilde{V}$ as well.

Proposition 2 *Let $\{\sigma, \hat{\sigma}, F, c\}$ be a sequential equilibrium. Then there exists a unique date $t = 0$ market equilibrium with $\{\sigma, \hat{\sigma}, F, c\}$.*

Proof. Given a SE, the price level is trivially given by (9). Private-sector bond holdings are given from the market clearing conditions, $\hat{b} = -b$ and $\hat{B} = -B$, and the lump-sum taxes are solved from the government budget constraints (10) and (11). The only non-trivial objects in a candidate AME are asset prices. The necessary first-order conditions associated with (12) are

$$\begin{aligned} u^c(c_0)\, q\left(s^t\right) &= \delta^t \mu\left(s^t|s_0\right) \int_{H^t} u^c\left(c\left(h^t\right)\right) F\left(dh^t|s^t\right), \\ u^c(c_0)\, Q\left(s^t\right) &= \delta^t \mu\left(s^t|s_0\right) \int_{H^t} \frac{u^c\left(c\left(h^t\right)\right)}{P\left(h^t\right)} F\left(dh^t|s^t\right) \end{aligned}$$

for all $s^t \in S^t$, $t \geq 1$. The only candidate for c_0 is y_0, as imposed by feasibility. Hence, for a given SE, asset prices are uniquely pinned down by the above necessary first-order conditions. Since $P(h^t) > 0$ for all $h^t \in H^t$, and $u^c(c) > 0$, asset prices are positive. Finally, consumption decisions satisfy the household budget constraints by Walras' Law ■

Asset prices in an AME are characterized by the necessary first-order conditions associated with (12). Since a SE requires $\pi_t = \hat{\pi}_t$ almost everywhere, $c(h^t) = y^*(s_t)$ with probability one along the equilibrium path. Therefore real and nominal asset prices satisfy

$$q\left(s^t\right) = \delta^t \mu\left(s^t|s_0\right)\frac{u^c\left(y^*\left(s_t\right)\right)}{u^c\left(y_0\right)}, \tag{15}$$

$$Q\left(s^t\right) = \delta^t \mu\left(s^t|s_0\right)\frac{u^c\left(y^*\left(s_t\right)\right)}{u^c\left(y_0\right)}\int_{H^t}\frac{1}{P\left(h^t\right)}F\left(dh^t|s^t\right) \tag{16}$$

for all $s^t \in S^t$, $t \geq 1$. Thus the real interest rate is exogenous. The price level $P(h^t)$ is a function only of π^t. This allows writing asset prices in terms of the associated sustainable plan M. It is useful to combine the asset price conditions:

$$Q\left(s^t\right) = q\left(s^t\right)\int_{\Pi^t}\frac{1}{P\left(h^t\right)}M\left(d\pi^t|s^t\right). \tag{17}$$

Finally I price a short-term nominal bond. It pays one nominal unit in all states of the world at date $t = 1$, and thus its discount price is

$$R_0^{-1} = \sum_{s_1 \in S} Q(s_1).$$

The two key outputs from an AME are the sustainable plan and the asset prices. I want to analyze the mapping between these two equilibrium objects. It turns out to be convenient to think of the mapping in terms of value sets. Let

$$\Gamma\left(\{q, Q\}\right) = \{v\left(c; F\right) | \{F, c, q, Q\} \text{ are a part of am AME}\}$$

defined over any positive asset price schedule $\{q, Q\}$. Proposition 2 has established that, given a sustainable plan M, there exists a unique set of asset prices that constitute an AME generating M. However, the converse is not true. Given asset prices $\{q, Q, \}$ there may be no SE conforming to an AME. Or there may be multiple SE. This is important in order to understand the results on coordination.

First I illustrate that there may be no AME for given asset prices $\{q, Q\}$. An extreme example is convenient. Assume there is a unique one-period Nash equilibrium, $\tilde{\pi}$, and, for simplicity, assume it is state invariant. It can be shown that, for low enough $\delta > 0$, only $\tilde{\pi}$

is a sustainable action and hence all SEs must feature $\tilde{\pi}$ with probability one. Therefore, for any asset prices $\{q, Q\}$ such that

$$Q\left(s^t\right) \neq q\left(s^t\right) \frac{1}{\tilde{\pi}^t}$$

there would be no AME with such prices. In this example, $\Gamma\left(\{q, Q\}\right) = \varnothing$ for all but one price schedule.

Next I consider the possibility of multiple SEs associated with given asset prices $\{q, Q\}$. For simplicity there is no exogenous uncertainty $s_t = \bar{s}$ for all $t \geq 1$, with $y^*\left(\bar{s}\right) = y_0$. Assume that there exists a SE $\{\sigma, \hat{\sigma}, F, c\}$ that places probability one in history $\tilde{h}^t = \{\tilde{\pi}\}^t$ for all $t \geq 0$ and its value lies in the interior of $V\left(\bar{s}\right)$. Then there exists a SE $\{\sigma', \hat{\sigma}', F', c'\}$ that places equal probability to histories $\{\tilde{\pi} + \varepsilon_1, \tilde{\pi}, \tilde{\pi}, ..\}$ and $\{\tilde{\pi} - \varepsilon_2, \tilde{\pi}, \tilde{\pi}, ..\}$ with

$$\frac{1/2}{\tilde{\pi} + \varepsilon_1} + \frac{1/2}{\tilde{\pi} - \varepsilon_2} = \frac{1}{\tilde{\pi}}$$

and ε_1 small enough. That $\{\sigma', \hat{\sigma}', F', c'\}$ is indeed a SE follows from the fact that, for an arbitrarily small ε_1, both $\tilde{\pi} + \varepsilon_1$ and $\tilde{\pi} - \varepsilon_2$ are sustainable actions. As long as the costs of inflation $g\left(\pi\right)$ have some curvature, the two SEs have different ex-ante welfare.

By construction, both SEs have the same asset prices,

$$\begin{aligned} q\left(s\right) &= \delta, \\ Q\left(s\right) &= \delta \frac{1}{\tilde{\pi}} = \delta\left(\frac{1/2}{\tilde{\pi} + \varepsilon_1} + \frac{1/2}{\tilde{\pi} - \varepsilon_2}\right). \end{aligned}$$

Hence these asset prices $\{q, Q\}$ do not determine the ex-ante welfare level of an AME. Hence, $\Gamma\left(\{q, Q\}\right)$ is not a singleton.

Hence the mapping from date $t = 0$ market equilibrium asset prices to sustainable plans is a correspondence, $\Gamma : \{q, Q\} \rightrightarrows \{V, \emptyset\}$. However, asset prices are informative to some extent. A key observation is that the arbitrage condition (17) implies a one-to-one mapping between asset prices and the following moment of the distribution of the price level,

$$q\left(s^t\right) / Q\left(s^t\right) = \left[\int_{\Pi^t} \frac{1}{P\left(h^t\right)} M\left(d\pi^t | s^t\right)\right]^{-1}.$$

Many SEs or no SE may share this moment, yet it is unlikely that all SEs do. In particular, the moment implies that asset prices constrain the support of the compatible sustainable price levels. For example, a sustainable plan assigning probability one to prices strictly above or strictly below $q\left(s^t\right) / Q\left(s^t\right)$ will not satisfy the moment.

The next is the key result of the paper. As I just discussed, the mapping between asset prices and sustainable policy plans is generally a correspondence. However, the exception is of utmost interest: a date $t = 0$ market equilibrium features the best sustainable equilibrium if and only if the one-period nominal bond is priced at a certain discount rate.

Theorem 2 *A date $t = 0$ asset market equilibrium features a best sequential equilibrium if and only if*

$$R_0^{-1} = \delta \sum_{s_1 \in S} \mu (s_1|s_0) \frac{u^c (y^* (s_1))}{u^c (y_0)} \frac{1}{\pi^* (s_1)}. \tag{18}$$

Proof. If an AME is a BSE, then (16) implies $Q(s_1) = \delta\mu(s_1|s_0) \frac{u^c(y^*(s_1))}{u^c(y_0)} \frac{1}{\pi^*(s_1)}$ for each $s_1 \in S$ and (18) follows trivially.

Consider now a AME satisfying (18). The arbitrage condition (17) implies that

$$\sum_{s_1 \in S} Q(s_1) = \sum_{s_1 \in S} q(s_1) \left(\int_\Pi \frac{1}{P(h^1)} F(d\pi_1|s_1) \right)$$

and combined with (18)

$$\sum_{s_1 \in S} q(s_1) \left(\frac{1}{\pi^*(s_1)} - \int_\Pi \frac{1}{P(h^1)} F(d\pi_1|s_1) \right) = 0.$$

By the definition of π^*, $P(h^1) \geq \pi^*(s_1)$ for any SE. Therefore

$$\frac{1}{\pi^*(s_1)} - \int_\Pi \frac{1}{P(h^1)} F(d\pi_1|s_1) \geq 0$$

for each $s_1 \in S$. It follows that $P(h^1) = \pi^*(s_1)$ for all $s_1 \in S$. Finally, Condition 1 implies that Corollary 1 applies and the only SE compatible with (18) is the BSE ■

The first thing to note about Theorem 2 is that the price of a single asset for a single date is sufficient to pin down the BSE for all periods. This stands in contrast with practically any other *complete* price schedule, i.e., it is not possible to pin down uniquely other SEs even if all asset prices can be determined. This strong result is built on the properties of the BSE summarized in Proposition 1. In particular, the BSE belongs to the boundary of both the value set V and the set of sustainable actions. It is straightforward to show that the best sequential equilibrium requires the highest discount rate for the short-term nominal interest rate.

Corollary 2 *For any date $t = 0$ asset market equilibrium,*

$$R_0^{-1} \leq \delta \sum_{s_1 \in S} \mu(s_1|s_0) \frac{u^c(y^*(s_1))}{u^c(y_0)} \frac{1}{\pi^*(s_1)}$$

The proof of Theorem 2 applies almost step-by-step to other asset trade arrangements. If only the short-term nominal bond were available, then (18) is exactly the associated necessary first-order condition. Asset trade could also be sequential with no change in the logic of the result. Of course, if asset holdings do change the monetary authority's incentives, then asset prices would also bring cooperative considerations.

It is important to note that Condition 1 is in effect: Theorem 2 does not apply if the zero nominal interest rate bound is binding for the BSE at date $t = 1$. It is easy to find parameters such that (18) would imply a negative interest rate. Interestingly, setting $\sum_{s_1 \in S} Q(s_1) = 1$ does not solve the problem. Because I use Corollary 1 to prove the result, if the BSE is the optimal monetary policy, there could be other sustainable plans that start at the zero nominal interest rate level but inflation eventually takes off.

Finally I ask whether there is an equivalent result to Theorem 2 for inflation. From Corollary 1 it is straightforward that the BSE is the only sustainable equilibrium compatible with a state-contingent inflation vector, $\{\pi(s_1) : s_1 \in S\}$. That is, inflation must be specified for all possible realizations of the state of the economy in the first period of the economy. If inflation is specified as a range, or at some point in the medium term, there will be multiple sustainable inflation paths that are compatible, although they will be a small subset of all sustainable inflation paths.

6 Coordination

So far my results have just moved the coordination problem to the spot market for the short-term nominal bond. The question remains o how the central bank can achieve coordination in the spot market. Most central banks appear to be able to influence the short-end of the nominal yield curve.

Below I sketch how coordination may occur. I introduce fictitious play prior to the opening of date $t = 0$ asset markets. Agents learn which short-term nominal interest rate will prevail. A particular trading arrangement—loosely based on how the Federal Reserve auctions U.S. Treasuries—leads to coordination on the highest discount price for the short-term nominal bond—but only if initial beliefs are heterogeneous.[21] My modeling of this stage

[21] There is some evidence that auctions can lead to coordination. See Huyck et al. (1993), Crawford and Boseta (1998), and Janssen (2006).

is so simple that it is close to trivial. In particular, it effectively decouples the learning and coordination stages from the rest of the game. A fully satisfactory model of coordination in the short-term nominal bond market remains beyond the scope of this paper.

6.1 A simple round of fictitious play

There are N rounds of fictitious play in what could be called the "primary market" for the short-term nominal bond. In each round n the monetary authority allocates one unit of the nominal bond using a second-price sealed-bid auction (or Vickrey auction). The bond is sold at discount. The Vickrey auction ensures that agents bid their true valuations. In addition, it is remarkably similar to the actual bidding on U.S. Treasury securities.[22] Let χ_n^* denote the winning bid at round n.

Agents start with heterogeneous beliefs regarding which short-term nominal bond sustainable equilibrium will be observed. Let ξ_n^i be a probability measure over support $\mathcal{Q}$, spanning all the possible equilibrium values the short-term nominal interest rate can take in a date $t = 0$ market equilibrium.[23] Posterior ξ_n^i summarizes the beliefs of agent i after n rounds of play, with the prior ξ_0^i being exogenous.[24] Agents update their beliefs each round with the observed winning bid, χ_n^*, using Bayes' rule. The key property needed for the simple argument presented here is quite trivial: if $\chi_n^* \geq \chi_{n-1}^i \equiv \sum_{\mathcal{Q}} Q_0 \xi_{n-1}^i(Q_0)$, then $\chi_n^i \geq \chi_{n-1}^i$, with strict inequality sign if $\chi_n^* > \chi_{n-1}^i$. That is, if the winning bid is higher than expected, agents revise their beliefs to expect a higher price next round. Other belief-updating rules in the literature share this property.

Finally, I assume winning agents value price gains or losses linearly, so their payoff is $Q - \chi_n^*$. In line with the fictitious play literature, agents do not try to influence the future play of other agents. Thus, quite trivially, agent i on round n submits a bid equal to χ_{n-1}^i. Assuming a dense distribution of beliefs, the winning bid is given by $\chi_n^* = \sup_i \chi_{n-1}^i$.

Now I compare the outcome of the fictitious play under two different set of initial beliefs. In the first scenario, agents' initial are heterogeneous. More precisely, beliefs ξ_0^i span all possible probability measures over $\mathcal{Q}$ so there is absolutely no prior coordination. In this scenario the winning bid is $\sup \mathcal{Q}$ in every round. As N grows arbitrarily large, all beliefs converge to a mass point on $\sup \mathcal{Q}$—and thus to the best sequential equilibrium.[25]

[22] See Edwards (1997) for a description of open market operations by the Federal Reserve.

[23] For simplicity I abstract from exogenous shocks.

[24] To avoid convergence issues I actually assume that support $\mathbf{Q}$ is discrete, with grid points arbitrarily close. Technically it is also necessary that the prior be non-doctrinaire—i.e., ξ_0^i assigns a positive probability to all outcomes—but I will ignore this point for simplicity.

[25] Janssen (2006) also shows how auctions can act as coordination devices without entertaining heterogeneous beliefs. Players bid for the right to play a coordination game; the notion of forward induction leads

In the second scenario, beliefs are homogeneous, so $\xi_0^i = \xi_0$ for any agent i. The winning bid is given by χ_0 in all rounds, and there is no update whatsoever of beliefs. So unless prior beliefs were coordinated in the best sustainable equilibrium, a sub-optimal equilibrium will be observed.

Thus the outcome of this simple model is exactly what is needed to implement coordination without destroying reputation: starting from heterogeneous beliefs, agents coordinate on the best sustainable inflation path. Once beliefs are coordinated, though, it is not possible to manipulate private-sector expectations into a different equilibrium. The latter part is crucial: if reputation is a "free lunch" at all times, then it is useless.

Interestingly, the model does not presuppose that the monetary authority is aware of the theory developed here. It only needs to target the highest discount price for the short-term nominal securities.

7 Conclusions

This paper puts forward the hypothesis that the monetary authority can effectively coordinate on the best sustainable inflation path. Effectively the coordination problem is moved from the very large space of inflation paths to the spot market for the short-term nominal bond. There are good reasons to think the monetary authority can effectively coordinate expectations on the short-term nominal interest rate; in addition, doing so seems possible without undermining the punishment associated with a loss of reputation.

My analysis of the coordination mechanism for the short-term nominal interest rate, though, is limited. The coordination stage is modeled as fictitious play, so it is completely decoupled from the rest of the model. It is not clear how to embed learning dynamics in a subgame perfect equilibrium: the history-dependence requires agents to be quite sophisticated regarding the strategies. Unfortunately this limitation also implies that any discussion of the monetary policy framework must remain speculative for now.

to the Pareto-superior outcome being the unique equilibrium.

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